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2009 J. Phys. A: Math. Theor. 42 400301

(http://iopscience.iop.org/1751-8121/42/40/400301)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 1–3 doi:10.1088/1751-8113/42/40/400301

PREFACE

Nonlinearity and Geometry: connections with integrability

Geometric ideas are present in many areas of modern theoretical physics and they are usually associated with the presence of nonlinear phenomena. Integrable nonlinear systems play a prime role both in geometry itself and in nonlinear physics. One can mention general relativity, exact solutions of the Einstein equations, string theory, Yang–Mills theory, instantons, solitons in nonlinear optics and hydrodynamics, vortex dynamics, solvable models of statistical physics, deformation quantization, and many others. Soliton theory now forms a beautiful part of mathematics with very strong physical motivations and numerous applications. Interactions between mathematics and physics associated with integrability issues are very fruitful and stimulating. For instance, spectral theories of linear quantum mechanics turned out to be crucial for studying nonlinear integrable systems. The modern theory of integrable nonlinear partial differential and difference equations, or the 'theory of solitons', is deeply rooted in the achievements of outstanding geometers of the end of the 19th and the beginning of the 20th century, such as Luigi Bianchi (1856–1928) and Jean Gaston Darboux (1842–1917). Transformations of surfaces and explicit constructions developed by 'old' geometers were often rediscovered or reinterpreted in a modern framework. The great progress of recent years in so-called discrete geometry is certainly due to strong integrable motivations. A very remarkable feature of the results of the classical integrable geometry is the quite natural (although nontrivial) possibility of their discretization.

This special issue is dedicated to Jean Gaston Darboux and his pioneering role in the development of the geometric ideas of modern soliton theory. The most famous aspects of his work are probably Darboux transformations and triply orthogonal systems of surfaces, whose role in modern mathematical physics cannot be overestimated. Indeed, Darboux transformations play a central role in soliton theory unifying continuous, discrete and quantum integrable systems. Triply orthogonal coordinates proved to be of prime importance for the modern theory of Hamiltonian systems of hydrodynamic type and differential-geometric Poisson brackets, culminating in the construction of the rich and beautiful theory of Frobenius manifolds.

The idea for this special issue developed out of the Second Workshop on Nonlinearity and Geometry, a successful conference held in the Mathematical Research and Conference Center at Będlewo, Poland, 13–19 April 2008 (http://wmii.uwm.edu.pl/~doliwa/WNG-DDhtml). However, there was an open call for papers for this issue and all contributions were peer reviewed according to the standards of the journal and taking into account their relevance to the subject of the planned issue. Among the 30 listed authors, 16 attended the conference and the remaining 14 submitted their papers in answer to this open call.

The First School on Nolinearity and Geometry ('Bianchi Days') was organized by Antoni Sym and his students in 1995 at the Physics Faculty of Warsaw University, Poland. The proceedings of the workshop, edited by Daniel Wójcik and Jan Cieśliński, were published by Polish Scientific Publishers PWN (Warsaw, 1998). The Second Workshop ('Darboux Days') was organized in 2008 by Adam Doliwa and his coworkers, under the Honorary Chair of Antoni Sym, as a Banach Center Conference. Both workshops gathered around 50 participants. The purpose of these meetings was to bring together researchers with diverse backgrounds (e.g., mathematical physics and differential geometry), and to review the state of the art at the border between the two subjects: geometric inspirations in soliton theory and applications of soliton techniques in geometry. The format was designed to allow substantial time for interaction and research. The invited lectures were longer, intended to present the current trends and open problems in the fields, and to be accessible to younger researchers.

It is not out of place to recall that earlier the Institute of Theoretical of Physics of Warsaw University organized two, now legendary, Jadwisin Soliton Workshops (1977 and 1979); see the short note in *Physica D: Nonlinear Phenomena* (1980 vol. 1, issue 1, pp 159–163) written by Antoni Sym who was deeply engaged in the organization of these conferences. In scale and scope both Jadwisin workshops preceded a series of very successful NEEDS conferences. Among the celebrated participants of the Jadwisin meeetings one can find names of great importance for the history of soliton theory: Martin Kruskal, Norman Zabuski, Mark Ablowitz, David Kaup, Allan Newell, Vladimir Zakharov, Sergei Manakov, Francesco Calogero, Antonio Degasperis and Ryogo Hirota.

This special issue begins with an introductory historical article in which Antoni Sym presents the most important ideas in the scientific biography of Gaston Darboux. We encourage the readers discover the greatest (scientific!) love of Darboux. This is followed by five review papers.

M Błaszak and B M Szablikowski discuss the general R-matrix formalism for the construction of integrable systems with infinitely many degrees of freedom. The general theory is applied to several infinite-dimensional Lie algebras leading to new examples of dispersionless and dispersive (soliton) integrable field systems in 1+1 and 2+1 dimensions.

J L Cieśliński presents the Darboux–Bäcklund transformation for 1+1-dimensional integrable systems of PDEs. He compares existing approaches to the construction of multisoliton Darboux matrices, discusses the nonisospectral case and presents some new results on the linear and bilinear invariants of the Darboux–Bäcklund transformation.

M Dunajski presents twistor theory as a geometric tool for solving nonlinear differential equations. Many soliton equations admit twistor interpretation in terms of holomophic vector bundles. A different approach is provided for dispersionless equations. Some integrable systems still await successful application of the twistor approach. This review, although concerned with advanced differential geometry, is quite elementary and self-contained.

F Nijhoff, J Atkinson and J Hietarinta review the construction of soliton solutions for the KdV type lattice equations and derive N-soliton solutions for all lattice equations in the Adler–Bobenko–Suris list except for the generic elliptic case. The same problem is addressed in the contribution by J Hietarinta and D J Zhang based on the more traditional direct Hirota method. This leads to Casoratians and bilinear difference equations.

Regular contributions include the following.

H Baran and M Marvan launch a project to classify integrable classes of surfaces based on a novel deformation procedure of the equations of the embedding. This leads to a remarkable new integrable equation describing a class of Weingarten surfaces which seems to be overlooked in the literature.

A Doliwa shows that the τ-function of the quadrilateral lattice can be identified with the Fredholm determinant of the integral equation inverting the nonlocal problem. This result is expected because its continuous counterpart (the case of conjugate nets, Darboux equations and the multicomponent KP hierarchy) is already known. Here one can find an explicit proof.

P Gaillard and V B Matveev consider special reductions of the generic Darboux–Crum dressing procedure, leading to new formulas for Darboux–Pöschl–Teller potentials, their difference deformations and the related eigenfunctions.

A Gouberman and K Leschke develop the theory of (generalized) Darboux transformations for conformal immersions of a Riemann surface into the 4-sphere. Applying this construction to the Clifford torus, they obtain a family of Willmore tori parametrized by Pythagorean triples.

V Kiselev and J W van de Leur construct compatible nontrivial finite deformations of the Lie algebra structure in the symmetry algebra of the 3-component dispersionless Boussinesq-type system.

T E Kouloukas and V G Papageorgiou introduce a family of nonparametric Yang–Baxter maps obtained by re-factorization of matrix polynomials of first degree. These maps are Poisson with respect to the Sklyanin bracket, and their degenerations are connected to known integrable systems on quad-graphs.

S V Manakov and P M Santini apply a novel version of the inverse scattering transform based on Lax pairs in multidimensional commuting vector fields to the heavenly and Pavlov equations, establishing that their localized solutions evolve without breaking, and constructing the long-time behaviour of the corresponding Cauchy problems.

Discretizations of integrable geometric models depend heavily on the coordinates used. M Nieszporski and A Sym show how to discretize Bianchi surfaces (associated with an elliptic version of the Ernst equation) in arbitrary parametrization.

C Rogers and A Szereszewski study the Bäcklund transformation for L-isothermic surfaces in the original Bianchi formulation. They establish a connection between this transformation and a nonhomogeneous linear Schrödinger equation and construct a class of generalized Dupin cyclides.

W K Schief, A Szereszewski and C Rogers study a classical system of equilibrium equations for shell membranes. Various examples of viable membrane geometries lead to remarkable geometric configurations such as generalized Dupin cyclides and L-minimal surfaces.

A Sergyeyev constructs infinite hierarchies of nonlocal higher symmetries for the oriented associativity equations using the spectral problem. The hierarchies in question generalize those constructed by Chen, Kontsevich and Schwarz for the WDVV equations.

J Shiraishi and Y Tutiya study an integro-differential equation which generalizes the periodic intermediate long wave equation. The kernel of the singular integral involved is a second order difference of the Weierstrass ζ-function. Using Sato's formulation, the authors demonstrate the integrability of the equation in question, and construct some special solutions.

P H van der Kamp discusses general aspects of the Cauchy and Goursat problems for lattice equations focusing on their well-posedness, as well as on periodic and travelling wave reductions.

We would like to express sincere thanks to all contributors, editorial staff and all involved in compiling this special issue.

Jan L Cieśliński, Eugene V Ferapontov, Alexander V Kitaev and Jonathan J C Nimmo

Guest Editors